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Abstract

In this work we address the stock estimation problem for two fishery models. We show that a tool from nonlinear control theory called "observer" can be helpful to deal with the resource stock estimation in the field of renewable resource management. It is often difficult or expensive to measure all the state variables characterising the evolution of a given population system, therefore the question arises whether from the observation of certain indicators of the considered system, the whole state of the population system can be recovered or at least estimated. The goal of this paper is to show how some techniques of control theory can be applied for the approximate estimation of the unmeasurable state variables using only the observed data together with the dynamical model describing the evolution of the system. More precisely we shall consider two fishery models and we shall show how to built for each model an auxiliary dynamical system (the observer) that uses the available data (the total of caught fish) and which produces a dynamical estimation $\hat{x}(t)$ of the unmeasurable stock state $x(t)$. Moreover the convergence speed of $\hat{x}(t)$ towards $x(t)$ can be chosen.

Keywords: Fishery models, Stage-structured population models, Estimation, Harvested Fish Population, Observers.

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1 Introduction and a short survey of *observers* design

The stock estimation is one of the most important problem in fishery science. One can quote J.A. Gulland [17]: *A major emphasis in fishery science has been on the problems of estimating current and past level using catch levels and fishing effort data.*

To make a policy decision about the exploitation of renewable ressources, it is necessary to take into account the state of the resource stocks. This implies the need of a good estimate of the available resource. Mathematical models are more and more used to describe the evolution of biological systems. Here, we consider two mathematical models for fishery resources. The first one is a "stage structured" model [43, 44] that describes the dynamics of a population divided in stage-classes (according to age, length or weight) and submitted to the fishing action. The second model is a "global" model that describes the evolution of a fish population that can move between an area where it can be harvested and a reserve area where no fishing is allowed [9]. Both models are given by systems of differential equations of the form

$$\dot{x} = f(x, E), \tag{1}$$

where E is the fishing effort (it can be seen as a control or an input) and $x(t)$ is the state of the system at time t. The state variable $x(t)$ represents the density of the population or the number of individuals by stage. For both models, the state $x(t)$ is not available for measurement. In practice, the only available information at time t is the value of the captures: this means that one can measure the total catch at each time t. The value of the captures can be seen as the measurable output of system (1). The output is in general a function of the state variable and the input, that is, $y(t) = h(x(t), E)$.

Now assuming that (1) is a "good" model of the system under consideration, if it is possible to have the value of the state at some time t_0 then it is possible to compute $x(t)$ for all $t \geq t_0$ by integrating the differential equation with the initial condition $x(t_0)$. Unfortunately, it is often not possible to measure the whole state at a given time and therefore it is not possible to integrate the differential equation because one does not know an initial condition. One can only have a partial information of the state and this partial information is precisely given by $y(t)$ the output of the system. Therefore we shall show how to use this partial information $y(t)$ together with the given model in order to have a dynamical estimate $\hat{x}(t)$ of the real unknown state variable $x(t)$. This estimate will be produced by an auxiliary dynamical system which uses the information $y(t)$ provided by the system (1). This dynamical system is generally of the form

$$\dot{\hat{x}} = g(\hat{x}, E, y). \tag{2}$$

It can be represented by Schema 1 The estimate error is given by $e(t) = \hat{x}(t) - x(t)$ and it satisfies the following "error equation"

$$\dot{e} = g(\hat{x}, E, y) - f(x, E) \tag{3}$$

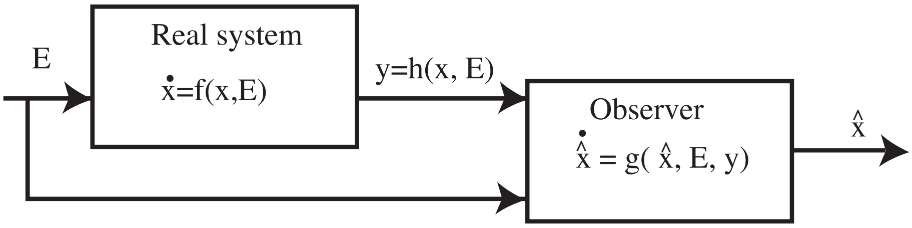

Figure 1: A schematic representation of an observer

The function g has to be determined in such a way that the solutions of (1) and (2) satisfy $x(t) - \hat{x}(t) \to 0$ as $t \to +\infty$ regardless of the respective initial conditions of system (1) and system (2).
A dynamical system (2) satisfying this conditions is called an "observer" for system (1). When the convergence of $\hat{x}(t)$ towards $x(t)$ is exponential, the system (2) is an "exponential observer". More precisely, system (2) is an exponential observer for system (1) if there exists $\lambda > 0$ such that, for all $t \geq 0$ and for all initial conditions $(x(0), \hat{x}(0))$, the corresponding solutions of (1) and (2) satisfy

$$\|\hat{x}(t) - x(t)\| \leq \exp(-\lambda t)\|\hat{x}(0) - x(0)\|.$$

In this situation a good estimate of the real unmeasured state is rapidly obtained. One must notice that we need not care about the choice of the initial condition of the observer since the convergence of $\hat{x}(t)$ towards the real state $x(t)$ does not depend on this choice.
When the system under consideration is a linear system, i.e., it can be written as follows

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \\ x(t) \in \mathbb{R}^n, \ u(t) \in U \subset \mathbb{R}^m, \ y(t) \in \mathbb{R}^q, \\ A, \ B, \text{ and } C \text{ are respectively } n \times n, \ n \times m \text{ and } q \times n \text{ matrices,} \end{cases} \tag{4}$$

then an exponential observer (called *Luenberger Observer*)[30] for this system is given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K\Big(y(t) - C\hat{x}(t)\Big) \tag{5}$$

where the $n \times q$ matrix K has to be computed. The Luenberger observer converges, i.e., $\mid \hat{x}(k) - x(k) \mid$ tends to zero exponentially fast if it is possible to find a matrix K in such a way that the eigenvalues of the matrix $A - KC$ are all with negative real part. It has been proved that such a matrix K exists if the pair (C, A) is observable.

The pair (C, A) is observable if and only if the matrix:

$$O_{(C,A)} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

is of rank n. In this case we say that the system (4), or the pair (C, A), satisfies the Kalman rank condition for observability (one can see for more details and examples [39] or [20]).

The construction of observers for highly nonlinear systems is still a very active research area in Control Theory. Several methods have been developed for some classes of systems (one can see for instance the references [30, 27, 28, 26, 47, 13] that represent different approaches). This is not an exhaustive list, because the literature on the subject is very extensive. This active research has resulted in the emergence of many nonlinear observer design techniques. The most classical one is based on the "feedback linearization" and the observer normal form (see for instance [6], [22], [27], [46]) Roughly speaking, this method consists in finding change of coordinates $x = \kappa(z)$, $u = \zeta(E)$, $y = \eta(w)$ in the state space as well as in the input space and in the output space in such a way that equation (1) is transformed into

$$\begin{cases} \dot{x} = & Ax + \chi(w, u), \\ w = & Cx. \end{cases} \tag{6}$$

In this case a Luenberger type observer can be easily constructed. However the conditions under which the appropriate changes of coordinates exist are restrictive. These changes of coordinates often exist only locally and hence the derived observer design works only locally.

The second famous method is the high gain construction ([41],[7],[13],[14], [15], [21], [4]). A short survey is given in [4]. This method is developed hereafter and will be used in this paper.

Another design method uses an on-line optimization approach ([24] , [2], [33], [34], [47]) such as moving horizon observers that use the integral output prediction error in the estimation process, and the observer using Newtons method. In this case, the state is estimated by minimizing a certain norm of the difference between the ob- server output and the measured output. The advantage of the online optimization method is the capability of dealing with a variety of nonlinear systems including time-varying systems, chaotic systems, and systems with unknown parameters. Moreover this method does not require the use of any canonical form. However, the corresponding observer computations are generally quite heavy and may prevent the use of these observers for systems with very fast dynamics.

Historically observability theory and observers design have been developed for artificial engineering systems but nowadays they are more and more applied to "natural systems". We outline here some applications of nonlinear observers to biological models. Once again the list is not exhaustive.

In [5] the well-known Droop model which describes the growth of a population of phytoplanktonic cells is considered. Observers for this model are built and are used to discuss the validity of this model by comparing the prediction of the state computed by the observer with direct measurements of this state.

In [10], observers are used to estimate the kinetic rates in bioreactors. The efficiency of the observer design is illustrated with examples dealing with the microbial growth and biosynthesis reactions.

A robust nonlinear asymptotic observer with adjustable convergence rate has been proposed in[1]. This observer has been applied to a model of an anaerobic digestion process used for wastewater treatment.

The authors of [29] consider a system of populations described by the classical Lotka-Volterra model with one predator and two preys. The only available information is the total quantity of population preys without distinction between them. An observer is constructed that allows to estimate all the state variables. It is also shown how the observer can be used for the estimation of the level of an abiotic effect on the population system. It must be, however, noticed that the proposed observer in [29] is a local observer, i.e., its convergence is guaranteed only if the initial estimate error is small.

A high gain observer is used in [42] to study a system describing a one-gene regulation circuit. The observer is used to to rebuild the non-measured concentrations of the mRNA and the protein.

The use of observer theory in fishery is scarce, we have done some works in this sense (see [35], [16]). In [35], an observer has been constructed for a stage structured discrete-time fishery model that exhibits an unknown recruitment function. In [16], a stage structured continuous model is considered and it is assumed that only the last class (mature individuals) is harvested. The present work is a continuation and a generalization of [16].

The goal of this paper is twofold. First we shall show that some tools from control theory are helpful to address the stock estimation problem for an exploited fish population. More precisely we shall built exponential observers for the two models under consideration. These observers will allow to give an estimate of the respective stocks. The second is to show that the application of mathematical tools to biological systems has to be done carefully. One of the most efficient way to build an observer for a nonlinear system has been given in [13]. We briefly recall the method developed in [13]. To simplify matters we consider systems without control. Roughly speaking, the result of [13] concerns systems that can be written (possibly after a coordinates change):

$$\left\{\begin{array}{l}\dot{z}(t)=\underbrace{\begin{pmatrix}0&1&0&\dots&0\\0&0&1&\dots&0\\.&.&.&.&.\\0&0&\dots&0&1\\0&0&\dots&0&0\end{pmatrix}}_{A}z(t)+\begin{pmatrix}0\\0\\\vdots\\0\\\psi(z(t))\end{pmatrix}=X(z(t))\\y(t)=z_1(t)=\underbrace{(1,0,\dots,O)}_{C}z(t).\end{array}\right. \tag{7}$$

The state of the system at time t is $z(t)=(z_1(t),z_2(t),\dots,z_n(t))\in\mathbb{R}^n$, and its measurable output is $y(t)$. The fact that $y(t)=z_1(t)$ means that one can measure only the first component of the state and hence the other components are not available for measurement. Assume that the function ψ is globally Lipschitz on $\mathbb{R}^n$, that is, there exists $K>0$ such that $|\psi(z)-\psi(x)|\le K|z-x|$ for all $(z,x)\in\mathbb{R}^n\times\mathbb{R}^n$. It has then been proved in [13] that for $\theta\ge 1$ large enough, an exponential observer (a *Luenberger type observer*) for the system (7) is given by the following dynamical system:

$$\dot{\hat{z}}=X(\hat{z})-S_\theta^{-1}C^T(C\hat{z}-y), \tag{8}$$

with S_θ being the solution of

$$\theta S_\theta+A^TS_\theta+S_\theta A=C^TC.$$

System (8) is an exponential observer for system (7) means that the solutions of (8) converge to the solutions of system (7) with an exponential speed regardless the values of the respective initial conditions $z(0)$ and $\hat{z}(0)$. To prove this result the authors of [13] use the fact that the function ψ is globally Lipschitz on the **whole** state space $\mathbb{R}^n$. The global Lipschitz assumption is very restrictive. Biological systems always evolve in a bounded domain $\mathcal{D}$ of $\mathbb{R}^n$ and hence the global Lipschitz assumption is satisfied on $\mathcal{D}$. However, it must be noticed that the fact that the domain $\mathcal{D}$ is positively invariant for system (7) and that the map ψ is globally Lipschitz on $\mathcal{D}$ does not guarantee the convergence of the observer (8) even if one take the initial values inside $\mathcal{D}$. Indeed, the domain $\mathcal{D}$ is positively invariant for the system (7) but it is **not** a positively invariant set for the system (8) defining the equations of the observer. More precisely, for a given initial condition $(z(0),\hat{z}(0))\in\mathcal{D}\times\mathcal{D}$, the corresponding solution $(z(t),\hat{z}(t))$ of (7-8) can leave the set $\mathcal{D}\times\mathcal{D}$ in finite time: the component $z(t)$ will actually belong to $\mathcal{D}$ for all positive time but there is no reason that the same property will be true for $\hat{z}(t)$. In order to built an exponential observer for the considered system in this situation, one has first to extend the function ψ from $\mathcal{D}$ to the whole $\mathbb{R}^n$ by a function $\tilde{\psi}$ which is globally Lipschitz on $\mathbb{R}^n$ and then to consider the systems (7-8) defined on $\mathbb{R}^n\times\mathbb{R}^n$ after replacing the function ψ by its prolongation $\tilde{\psi}$. The stage-structured fishery model we consider here will illustrate this fact. For this model, there is a domain $\mathcal{D}\subset\mathbb{R}^3$ which is positively invariant, and the system dynamics are defined by a vector field X which is globally Lipschitz on $\mathcal{D}$. We shall show that the observer works well when we extend the vector field X to the whole space $\mathbb{R}^3$ and it fails to work when the prolongation is not done. The same things are valid for the global model. This shows that the Lipschitz extension of the vector field mentioned in [13] is not only for mathematical sophistication purpose but it is also necessary for application purpose. Here we construct simply a continuous Lipschitz extension of the function ψ. For more details concerning the design of Lipschitz extensions one can see for instance [38].

The paper is organized as follows. In Section 2, we present the stage-structured model and we built an observer for this system. The construction is made for a three stages model. It can be done for an arbitrary number of stages but the calculus are longer and more complicated. Section 3 is devoted to the stock estimation problem for a "global" model. Once again, for clarity reasons, we have preferred to deal with a model with two fishing areas but the observer construction can be done for a system describing the dynamics of a fish population that can move between different fishing zones (an example of such a system has been considered in [32]).

2 A Stage-structured model

In this section, we consider a class of a structured model in fishery with three classes. The first class x_0 is constitute of the pre-recruits i.e the eggs, larvae and the juveniles. The second and the third classes are the post-recruits or the exploited phase of the population.

The dynamics of the system are modeled by the following three dimensional system (see [43, 44], [36]) :

$$\begin{cases} \dot{x}_0(t) = -\alpha_0 x_0(t) + \sum_{i=1}^{2} f_i l_i x_i(t) - \sum_{i=1}^{2} p_i x_i(t) x_0(t) - p_0 x_0^2(t) \\ \dot{x}_1(t) = \alpha x_0(t) - (\alpha_1 + q_1 E) x_1(t) \\ \dot{x}_2(t) = \alpha x_1(t) - (\alpha_2 + q_2 E) x_2(t) \end{cases} \tag{9}$$

where :

x_i: the number of fish in the stage i.
α : linear aging coefficient (in time^{-1})
m_i : natural mortality rate of class i (in time^{-1})
$\alpha_i = m_i + \alpha$ (in time^{-1})
p_0: juvenile competition parameter (in time^{-1}.number^{-1})
f_i: fecundity rate of class i (no dimension)
l_i: reproduction efficiency of class i (in time^{-1})
p_i: predation rate of class i on class 0 (time^{-1}.num^{-1})
q_i: capturability coefficient of class i (in unit effort^{-1})
E: instantaneous fishing effort. (in unit effort$\times$time^{-1}).

We assume that the total catch is available for measurement. This total catch can be considered as a measurable output of the system(9) and it is given by

$$y(t) = q_1 E x_1(t) + q_2 E x_2(t) \tag{10}$$

We then obtain the following coupled system:

$$\begin{cases} \dot{x}_0(t) = -\alpha_0 x_0(t) + \sum_{i=1}^{2} f_i l_i x_i(t) - \sum_{i=1}^{2} p_i x_i(t) x_0(t) - p_0 x_0^2(t) \\ \dot{x}_1(t) = \alpha x_0(t) - (\alpha_1 + q_1 E) x_1(t) \\ \dot{x}_2(t) = \alpha x_1(t) - (\alpha_2 + q_2 E) x_2(t) \\ \\ y(t) = q_1 E x_1(t) + q_2 E x_2(t) \end{cases} \tag{11}$$

We consider system (11) which is a nonlinear system. Our aim is to construct an observer (estimator) i.e an auxiliary system which will give a dynamical estimate $(\hat{x}_0(t), \hat{x}_1(t), \hat{x}_2(t))$ of the state $(x_0(t), x_1(t), x_2(t))$ of system (9). For the construction of such auxiliary system, we shall use a method called High Gain construction (see for instance [13]). This construction provide an exponential observer; the estimation error will converges to zero with exponential speed, i.e.,

$$\|\hat{x}(t) - x(t)\| \leq \exp(-\lambda t)\|\hat{x}(0) - x(0)\|.$$

2.1 High Gain observer design for (11)

The system (11) is the system (9) coupled with the output (10). For the observer design, we will use the High Gain observer techniques (Gauthier et al.([13])) to construct a High Gain observer for system (9).
It has been proved in [43] that there is a positively invariant compact set for system (9). This set is of the form $D = [a_0, b_0] \times [a_1, b_1] \times [a_2, b_2]$, where the numbers a_i can be chosen as small as we need and the numbers b_i are function of the parameters f_i, l_i and p_i. More precisely:

$$\begin{aligned} & b_i = \pi_i \mu \\ & \text{with } \pi_i = \frac{\alpha^i}{\prod_{j=1}^{i}(\alpha_j + q_j E)}, \\ & \text{and } \mu = \min_{i\,:p_i \neq 0} \{\frac{f_i l_i}{p_i}\} \end{aligned}$$

Let us denote by F the vector field defining the dynamics of the system (9), and h the output function, that is $y(t) = h(x(t)) = q_1 E x_1(t) + q_2 E x_2(t)$ and

$$F(x(t)) = \begin{pmatrix} -\alpha_0 x_0(t) + \sum_{i=1}^{2} f_i l_i x_i(t) - \sum_{i=1}^{2} p_i x_i(t) x_0(t) - p_0 x_0^2(t) \\ \alpha x_0(t) - (\alpha_1 + q_1 E) x_1(t) \\ \alpha x_1(t) - (\alpha_2 + q_2 E) x_2(t) \end{pmatrix}$$

Let Φ be the function $\Phi : \overset{o}{D} \to \mathbb{R}^3$ ($\overset{o}{D}$ is the interior of D), defined as follows:

$\Phi(x) = \begin{pmatrix} h(x) \\ L_F h(x) \\ L_F^2 h(x) \end{pmatrix}$, where L denotes the Lie derivative operator with respect to the vector field F. Thus,

$$\Phi(x) = E \begin{pmatrix} q_1 x_1 + q_2 x_2 \\ \alpha q_1 x_0 + \Big(\alpha q_2 - q_1(\alpha_1 + q_1 E)\Big) x_1 - q_2(\alpha_2 + q_2 E) x_2 \\ \begin{array}{c} \Big(- \alpha_0 \alpha q_1 + \alpha^2 q_2 - \alpha q_1(\alpha_1 + q_1 E)\Big) x_0 \\ + \Big(\alpha q_1 f_1 l_1 - \alpha q_2(\alpha_1 + q_1 E) + q_1(\alpha_1 + q_1 E)^2 - \alpha q_2(\alpha_2 + q_2 E)\Big) x_1 \\ + \Big(\alpha q_1 f_2 l_2 + q_2(\alpha_2 + q_2 E)^2\Big) x_2 \\ -\alpha q_1 p_0 \, x_0^2 - \alpha q_1 p_1 \, x_1 x_0 - \alpha q_1 p_2 \, x_2 x_0 \end{array} \end{pmatrix}$$

The Jacobian of Φ can be written:

$$\frac{d\Phi}{dx} = \begin{pmatrix} 0 & q_1 E & q_2 E \\ \alpha q_1 E & \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 & \gamma_5 \end{pmatrix},$$

and

$$\Big[\frac{d\Phi}{dx}\Big]^{-1} = \frac{1}{\Gamma} \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_4 & \beta_5 & \beta_6 \\ \beta_7 & \beta_8 & \beta_9 \end{pmatrix},$$

where:

$\Gamma = \mathrm{Det}\Big(\frac{d\Phi}{dx}\Big) = q_1 E \gamma_2 \gamma_3 + \alpha q_1 q_2 E^2 \gamma_4 - q_2 E \gamma_1 \gamma_3 - \alpha q_1^2 E^2 \gamma_5$

$\gamma_1 = \alpha q_2 E - q_1 E(\alpha_1 + q_1 E)$

$\gamma_2 = -q_2 E(\alpha_2 + q_2 E)$

$\gamma_3 = \alpha^2 q_2 E - \alpha_0 \alpha q_1 E - \alpha q_1 E(\alpha_1 + q_1 E) - 2\alpha q_1 E p_0 x_0 - \alpha q_1 E p_1 x_1 - \alpha q_1 E p_2 x_2$

$\gamma_4 = q_1 E(\alpha_1 + q_1 E)^2 - \alpha q_2 E(\alpha_1 + q_1 E) - \alpha q_2 E(\alpha_2 + q_2 E) + \alpha q_1 f_1 l_1 E - \alpha q_1 E p_1 x_0$

$\gamma_5 = q_2 E(\alpha_2 + q_2 E)^2 + \alpha q_1 f_2 l_2 E - \alpha q_1 E p_2 x_0$

$\beta_1 = \gamma_1 \gamma_5 - \gamma_2 \gamma_4$

$\beta_2 = -q_1 E \gamma_5 + q_2 E \gamma_4$

$\beta_3 = q_1 E \gamma_2 - q_2 E \gamma_1$

$\beta_4 = -\alpha q_1 E \gamma_5 + \gamma_2 \gamma_3$

$\beta_5 = -q_2 E \gamma_3$

$\beta_6 = \alpha q_1 q_2 E^2$

$\beta_7 = \alpha q_1 E \gamma_4 - \gamma_1 \gamma_3$

$\beta_8 = q_1 E \gamma_3$

$\beta_9 = -\alpha q_1^2 E^2$.

The determinant of $\dfrac{d\Phi}{dx}$ can be written

$$\Gamma(x_0, x_1, x_2) = \mathrm{Det}\Big(\frac{d\Phi}{dx}\Big) = (c + a_0 x_0 + a_1 x_1 + a_2 x_2)\, E^3,$$

where c and a_i are functions of the parameters. The map $(x_0, x_1, x_2) \mapsto \Gamma(x_0, x_1, x_2)$ is affine on the polyhedron D, hence it reaches its extrema on the vertexes of D. For a given set of parameters, it is then sufficient to compute the values of $\Gamma(x_0, x_1, x_2)$ on the vertexes of D in order to see if $\Gamma(x_0, x_1, x_2)$ vanishes in D or not.
We assume that the parameters are such that the map Φ is a diffeomorphism from $\overset{o}{D}$ to $\Phi(\overset{o}{D})$. This implies that system (11) is observable.
In the new coordinates defined by $(z_1, z_2, z_3)^T = z = \Phi(x){=}(h(x), L_F h(x), L_F^2(x))^T$, our system can be written in the canonical form as follow:

$$\left\{\begin{array}{l} \dot z(t) = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{A} z(t) + \begin{pmatrix} 0 \\ 0 \\ \psi(z(t)) \end{pmatrix} \\ y(t) = z_1(t) = \underbrace{(1, 0, O)}_{C}\, z(t). \end{array}\right. \tag{12}$$

where : $\psi(z) = L_F^3 h(\Phi^{-1}(z)) = L_F^3 h(x) = \varphi(x)$

The function φ is smooth (it is a polynomial function of $x = (x_0, x_1, x_2)$) on the compact set D. Hence, it is globally Lipschitz on D. Therefore it can be extended by $\tilde\varphi$, a Lipschitz function on $\mathbb{R}^3$ which satisfies $\tilde\varphi(x) = \varphi(x)$, for all $x \in D$. In the same way we define $\tilde\psi$ the Lipschitz prolongation of the function ψ.
So we have the following system (13) defined on the whole space $\mathbb{R}^3$. The restriction of (13) to the domain D is the system (12):

$$\left\{\begin{array}{l} \dot z = Az + \begin{pmatrix} 0 \\ 0 \\ \tilde\psi(z) \end{pmatrix}, \\ y = Cz. \end{array}\right. \tag{13}$$

Hence, we have shown that system (11) satisfies the conditions of the following result which provides the observer construction.

Proposition 2.1 *([13]) Under the assumptions that*

H1: *Φ is a diffeomorphism from $\overset{o}{D}$ to $\Phi(\overset{o}{D})$. ($\overset{o}{D}$ is the interior of D).*

H2: *φ can be extended from D to $\mathbb{R}^3$ by a C^∞ function, globally Lipschitz on $\mathbb{R}^3$.*

Then an exponential observer for system (13) is given by the following system :

$$\dot{\hat{z}} \;=\; A\hat{z} + \psi(\hat{z}) + S^{-1}(\theta)C^T(y - C\hat{z}). \tag{14}$$

where $S(\theta)$ is the solution of

$$0 = -\theta S(\theta) - A^T S(\theta) - S(\theta)A^T + C^T C,$$

and θ is large enough.

Here, $S(\theta) = \begin{pmatrix} \theta^{-1} & -\theta^{-2} & \theta^{-3} \\ -\theta^{-2} & 2\,\theta^{-3} & -3\,\theta^{-4} \\ \theta^{-3} & -3\,\theta^{-4} & 6\,\theta^{-5} \end{pmatrix}$.

Precisely $\theta \geq 2ncK\sqrt{S}$, where K is the lipschitz coefficient of the function ψ, n is the dimension of the space, and $S = sup_{i,j}|S(1)_{i,j}|$.
For the proof one can see [13].
Going back to the our original system (9) via the transformation Φ^{-1}, we have :

$$\dot{\hat{x}} = \tilde{F}(\hat{x}) + \left[\frac{d\Phi}{dx}\right]^{-1}_{x=\hat{x}} \times S(\theta)^{-1}C^T(y - h(\hat{x})) \tag{15}$$

The restriction of this system to D is the following system :

$$\left\{\begin{array}{rl} \dot{\hat{x}}_0 = & -\alpha_0\hat{x}_0 + \displaystyle\sum_{i=1}^{2} f_i l_i \hat{x}_i - \sum_{i=1}^{2} p_i\hat{x}_i\hat{x}_0 - p_0\hat{x}_0^2 \\ & +(3\theta\beta_1 + 3\theta^2\beta_2 + \theta^3\beta_3)(y - q_1E\hat{x}_1 - q_2E\hat{x}_2) \\ \dot{\hat{x}}_1 = & \alpha\hat{x}_0 - (\alpha_1 + q_1E)\hat{x}_1 \\ & +(3\theta\beta_4 + 3\theta^2\beta_5 + \theta^3\beta_6)(y - q_1E\hat{x}_1 - q_2E\hat{x}_2) \\ \dot{\hat{x}}_2 = & \alpha\hat{x}_1 - (\alpha_2 + q_2E)\hat{x}_2 \\ & +(3\theta\beta_7 + 3\theta^2\beta_8 + \theta^3\beta_9)(y - q_1E\hat{x}_1 - q_2E\hat{x}_2) \end{array}\right. \tag{16}$$

which is the observer for the fishery model (9). This observer is particularly simple since it is only a copy of (9), together with a corrective term depending on θ.

2.2 Simulations and comments

We present here some simulation results that show the efficiency of the observer of system (9). The simulations have been done with the free software SCILAB.

Remarque 2.1 *For the simulations we extend the function φ by continuity in order to make it globally lipschitz on $\mathbb{R}^3$ in the following way: We denote $\tilde{\varphi}$ the prolongation of φ to $\mathbb{R}^3$ and the function π the projection on the domain D and we construct $\tilde{\varphi} = \varphi \circ \pi$. The extended function $\tilde{\varphi}$ has the same Lipschitz coefficient as φ. The projection π is defined as follows: for $x \in \mathbb{R}^3$, $\pi(x) = \bar{x}$, where $\bar{x} \in D$ is such that $dist(x, D) = \|x - \bar{x}\|$, i.e., $\bar{x}$ satisfies $\|x - \bar{x}\| = \min_{u\in D} \|u - x\|$. The extension algorithm is described in Appendix B..*

We use the following fishery parameters [36], [43].
$\alpha_0 = 1.3$; $\alpha_1 = 0.9$;
$\alpha_2 = 0.85$; $p_0 = 0.2$;
$p_1 = 0.1$; $p_2 = 0.1$;
$q_1 = 0.07$; $q_2 = 0.15$;
$f_1 = 0.5$; $f_2 = 0.5$;
$l_1 = 10$; $l_2 = 10$;
$E = 0.5$; $\alpha = 0.8$.

For these parameter the Jacobian of the function Φ is expressed as:

$$\frac{d\Phi}{dx} = \begin{pmatrix} 0 & 0.035 & 0.075 \\ 0.028 & 0.027275 & -0.069375 \\ \begin{array}{c} -0.01458 - 0.0112x_0 \\ -0.0028x_1 - 0.0028x_2 \end{array} & 0.0589979{-}0.0028x_0 & 0.191338{-}0.0028x_0 \end{pmatrix}$$

The determinant of this matrix is:

$$\mathrm{Det}(\frac{d\Phi}{dx}) = 1.612 \times 10^{-6} + 0.00004697x_0 + 0.0000125265x_1 + 0.0000125265x_2.$$

The states x_0, x_1 and x_2 are time varying but remain in the positive orthant; so the $\mathrm{Det}(\frac{d\Phi}{dx})$ does not vanish. Therefore $\frac{d\Phi}{dx}$ is invertible and then $\Phi(x)$ is a diffeomorphism.

With the parameters defined in the top of this section, we compute the coordinates of the higher corner B of the parallelepiped D ([43]) and we get $B = (25; 20.639; 17.868)$.
The nontrivial equilibrium point is $x^* = (18.572; 15.89; 13.743)$.

The construction of the high gain observer (15) is done with $\theta = 17$. For the simulations we have taken $x(0) = [21; 20; 15]$ and $\hat{x}(0) = [35; 40; 10]$.

Comments: Using the same parameters values, when we do not use the Lipschitz prolongation of the function φ to the whole $\mathbb{R}^3$, the state estimation $\hat{x}(t)$ computed by the observer tends to infinity in finite time. This actually happens in the beginning of the integration process as it can be seen in Figures 2, 4 and 6. When the Lipschitz prolongation of the function φ to the whole $\mathbb{R}^3$ is done, the convergence of the estimates delivered by the observer is quite fast (Figures 3, 5 and 7).

3 A global model

3.1 The model and the observer

Here we consider the dynamics of a fish population moving between two zones (see [9]). The first zone is a free fishing area, and the second zone is a reserve area where no fishing is allowed. Let $x_1(t)$ be the biomass density at time t of the fish population in the free fishing area and $x_2(t)$ be the biomass density at a time t of the fish population in the reserved areas. For $(i,j) \in \{1,2\}^2$, we denote by m_{ij} the migration rate from the zone i to the zone j. In the free fishing area, the total fishing effort is denoted by E. The growth of the two sub-population in each zone follows logistic model. The dynamics of the fish subpopulations in unreserved and reserved ares are then assumed to be governed by the following autonomous system of differential equations [9].

$$\begin{cases} \dot{x}_1 = r_1 x_1 \left(1 - \dfrac{x_1}{K_1}\right) - m_{12}x_1 + m_{21}x_2 - qEx_1 \\ \dot{x}_2 = r_2 x_2 \left(1 - \dfrac{x_2}{K_2}\right) + m_{12}x_1 - m_{21}x_2. \end{cases} \tag{17}$$

r_1 and r_2 represent the intrinsic growth of each fish sub-population, respectively, K_1 and K_2 are the carrying capacities of fish species in the unreserved and reserved areas, respectively; q is the catchability coefficient of fish species in the unreserved area. The parameters r_1, r_2, q, m_{12}, m_{21}, K_1 and K_2 are positives constants.
To the system (17) we associate the capture (i.e. the output) $y = qEx_1$ (the total of caught fish in the unreserved area), with this output, we show the observability condition of system (17) and construct an auxiliary system that will give a dynamical estimation of the state of system (17).
It is possible to find a positive real number w_0 in such a way that for any $w \geq w_0$ the following compact set D_w is positively invariant for system (17). This compact set is given

$$D_w = \{(x_1, x_2) \in \mathbb{R}_+^2 \ : \ x_1 + x_2 \leq w\},$$

The proof of this fact as well as the computation of w_0 as a function of the parameters are given in Appendix A..
Let us denote by f the vector field that defines the system (17):

$$f(x) = \begin{pmatrix} r_1 x_1 \left(1 - \dfrac{x_1}{K_1}\right) - (m_{12} + qE)x_1 + m_{21}x_2 \\ r_2 x_2 \left(1 - \dfrac{x_2}{K_2}\right) + m_{12}x_1 - m_{21}x_2 \end{pmatrix}.$$

Let $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $y(t) = h(x) = qEx_1(t)$ and

$$\Phi(x) = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} qEx_1 \\ r_1 qEx_1 \left(1 - \dfrac{x_1}{K_1}\right) - (m_{12} + qE)qEx_1 + m_{21}qEx_2 \end{pmatrix}.$$

Therefore $\dfrac{d\Phi}{dx} = \begin{pmatrix} qE & 0 \\ r_1 qE - \dfrac{2r_1 qEx_1}{K_1} - (m_{12} + qE)qE & m_{21}\, qE \end{pmatrix}$

and $\text{Det}\left(\frac{d\Phi}{dx}\right) = q^2E^2m_{21}$.

As the parameters q, E and m_{21} are positive ($\neq 0$), we can conclude that $\text{Det}\left(\frac{d\Phi}{dx}\right) \neq 0$, and then, Φ is a diffeomorphism from $\mathbb{R}^2$ to $\Phi(\mathbb{R}^2)$, thus system (17) is observable. Thanks to ([13]) the observer can be expressed as follows:

$$\dot{\hat{x}} = \tilde{f}(\hat{x}) + \left(\frac{d\Phi}{dx}\right)^{-1} \times S(\theta)^{-1}C^T(y - h(\hat{x})), \tag{18}$$

where $\tilde{f}$ is a Lipschitz extension of the function f from the invariant domain D_w to the whole $\mathbb{R}^2$ space, $C = (1, 0)$ and

$S(\theta)^{-1} = \begin{pmatrix} 2\theta & \theta^2 \\ \theta^2 & \theta^3 \end{pmatrix}$, with $\theta \geq 1$.

The restriction of the estimator (18) to the invariant domain D_w is given by the equations:

$$\begin{cases} \dot{\hat{x}}_1 = r_1\hat{x}_1\left(1 - \frac{\hat{x}_1}{K_1}\right) - m_{12}\hat{x}_1 + m_{21}\hat{x}_2 - qE\hat{x}_1 + 2\theta(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 = r_2\hat{x}_2\left(1 - \frac{\hat{x}_2}{K_2}\right) + m_{12}\hat{x}_1 - m_{21}\hat{x}_2 \\ \qquad\qquad +2\theta(\frac{qE}{m_{21}} - \frac{1}{m_{21}} + \frac{m_{12}}{qEm_{21}} + \frac{2r_1x_1}{m_{21}} + \frac{\theta}{m_{21}})(x_1 - \hat{x}_1), \end{cases} \tag{19}$$

3.2 Simulation

Simulations for the model (17) together with its observer (18) have been done with the following parameters :

$r_1 = \frac{7}{10}$; $r_2 = \frac{5}{10}$,

$q = \frac{25}{100}$, $E = \frac{9}{10}$,

$K_1 = 10$, $K_2 = \frac{22}{10}$,

$m_{12} = \frac{2}{10}$, $m_{21} = \frac{1}{10}$,

Thanks to formula (20) we compute $w_0 = 8.987$ and we take $w = 20$. With these parameters, the invariant domain is the triangle defined by $O(0, 0)$, $A(w, 0) = A(20, 0)$ and $B(0, w) = B(0, 20)$, and we take $\theta = 4$.

Using the SCILAB free software, the time evolution of the states as well as the respective estimates when the Lipschitz extension is done are drawn in Figures 8 and 10. When the Lipschitz extension has not been done, the simulations are given in Figures 9 and 11.

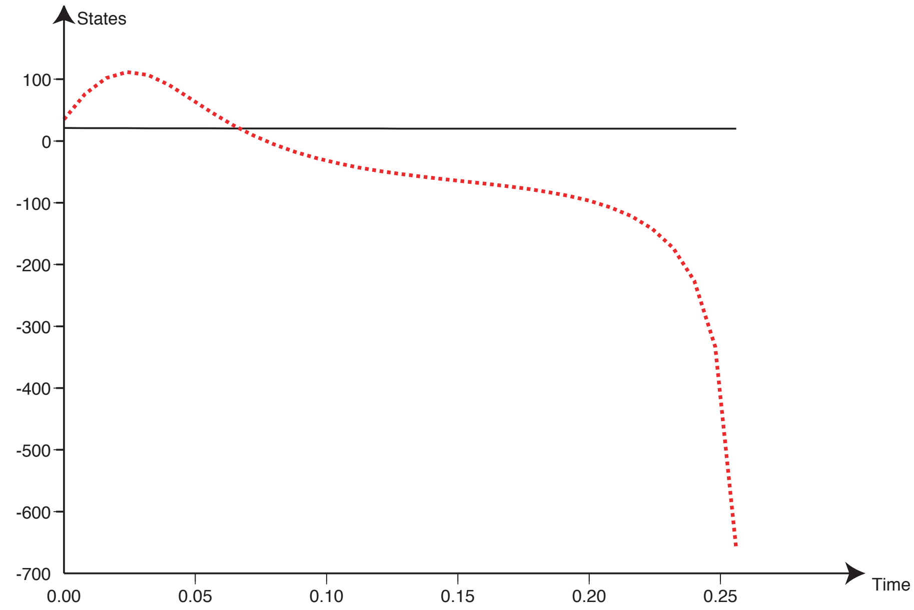

Figure 2: Simulation of system (9) with its observer (15): x_0 (solid line) and its estimate $\hat{x}_0$ (dashed line) when φ is not extended

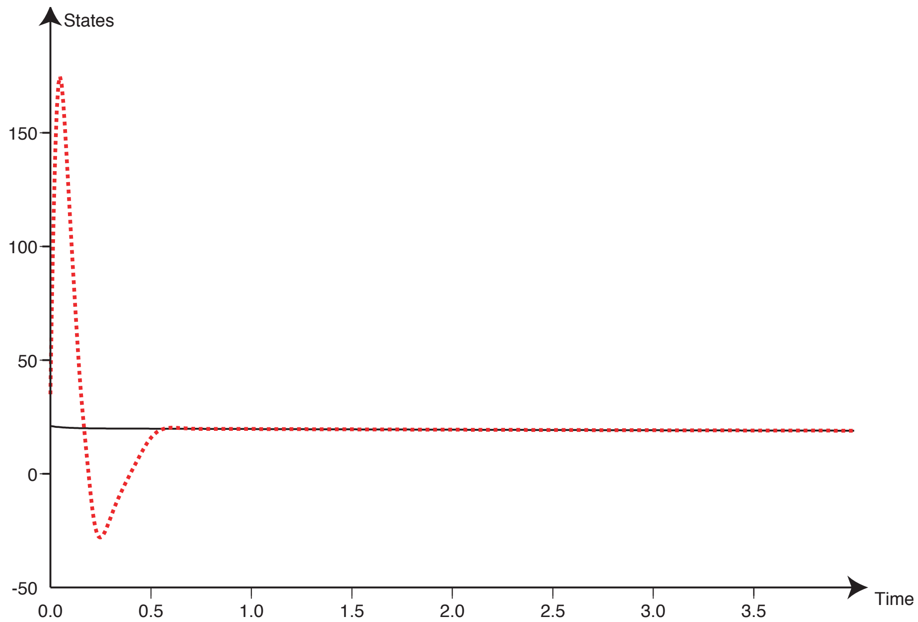

Figure 3: Simulation of system (9) with its observer (15): x_0 (solid line) and its estimate $\hat{x}_0$ (dashed line) when φ is extended

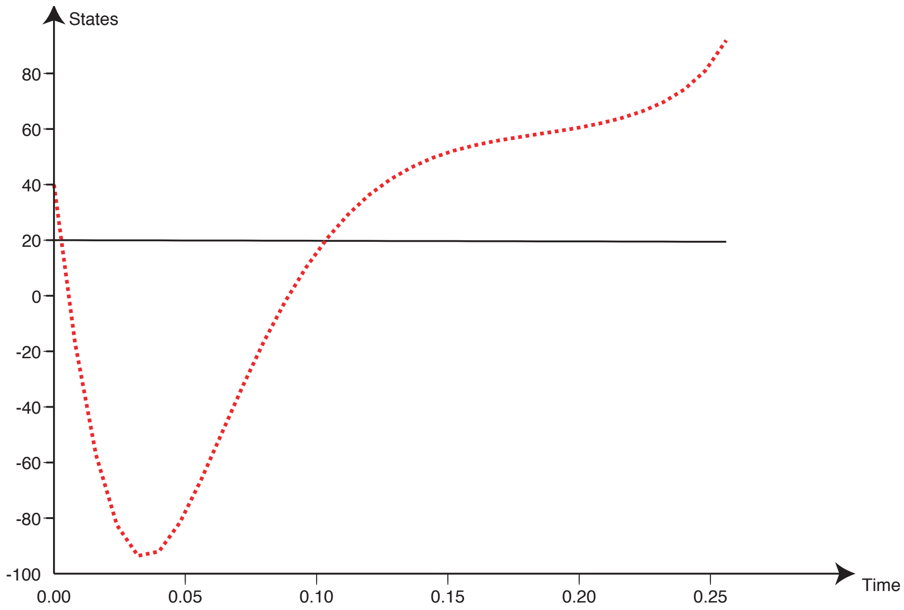

Figure 4: Simulation of system (9) with its observer (15): x_1 (solid line) and its estimate $\hat{x}_1$ (dashed line) when φ is not extended

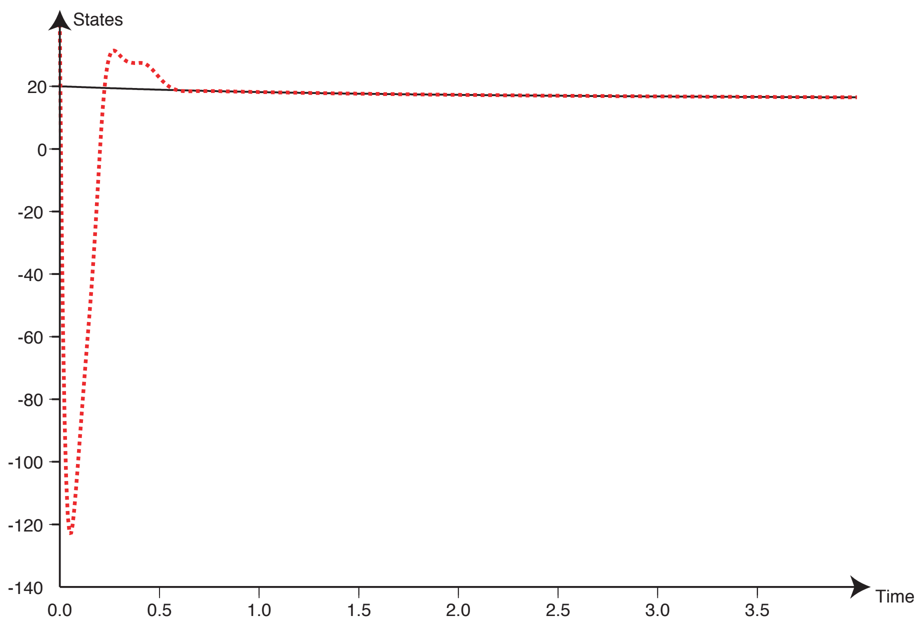

Figure 5: Simulation of system (9) with its observer (15): x_1 (solid line) and its estimate $\hat{x}_1$ (dashed line) when φ is extended

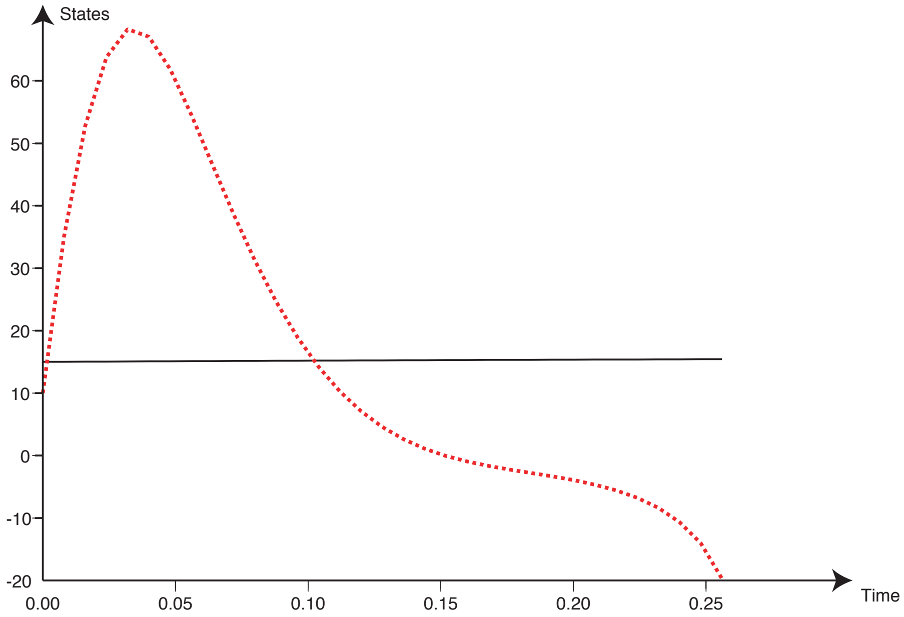

Figure 6: Simulation of system (9) with its observer (15): x_2 (solid line) and its estimate $\hat{x}_2$ (dashed line) when φ is not extended

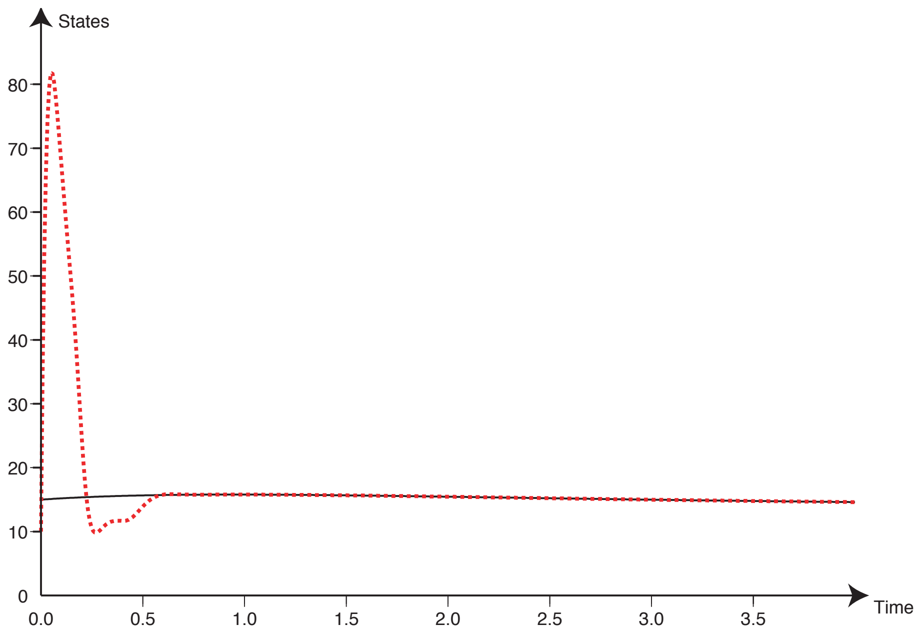

Figure 7: Simulation of system (9) with its observer (15): x_2 (solid line) and its estimate $\hat{x}_2$ (dashed line) when φ is extended

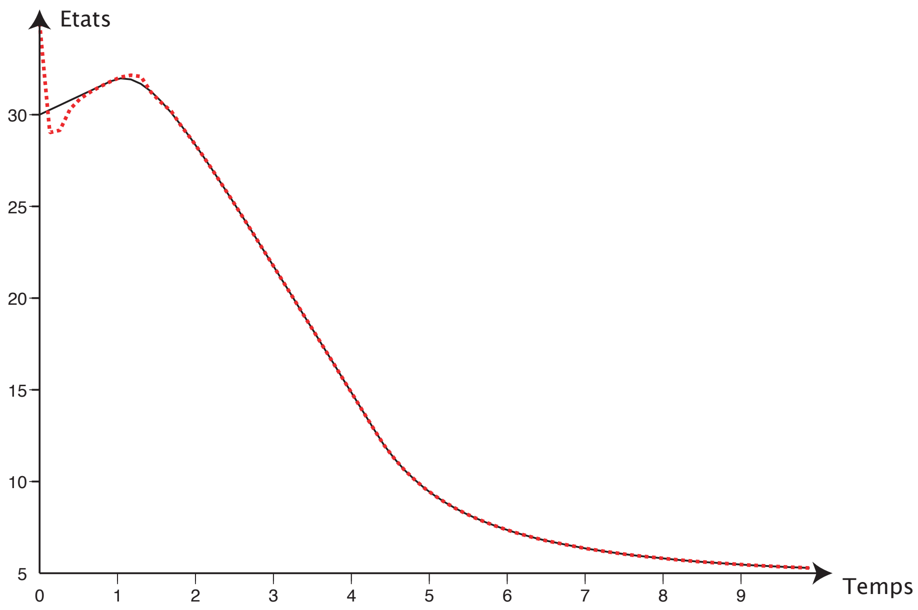

Figure 8: Simulation of system (17) with its observer (18): x_1 (solid line) and its estimate $\hat{x}_1$ (dashed line) when f is extended

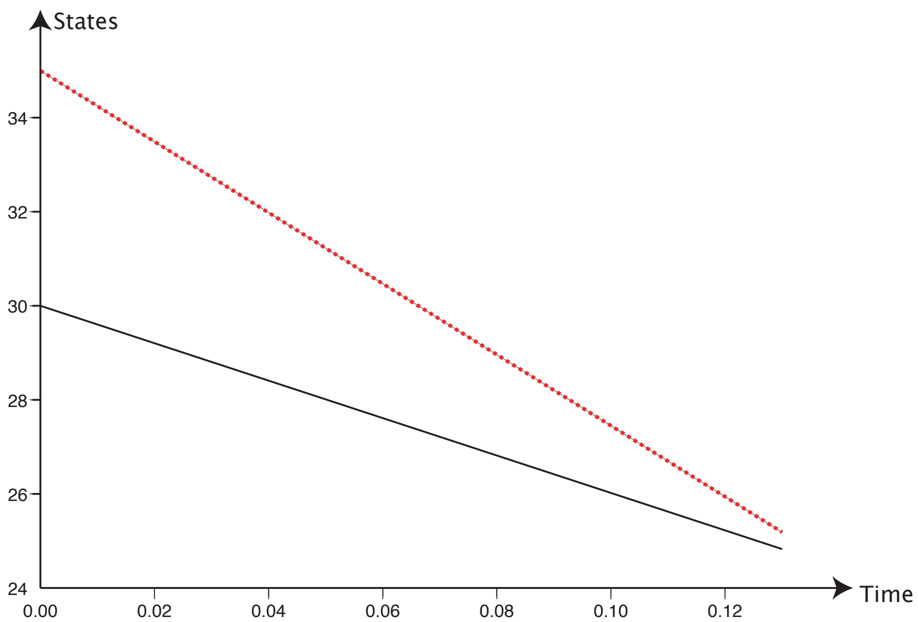

Figure 9: Simulation of system (17) with its observer (18): x_1 (solid line) and its estimate $\hat{x}_1$ (dashed line) when f is not extended

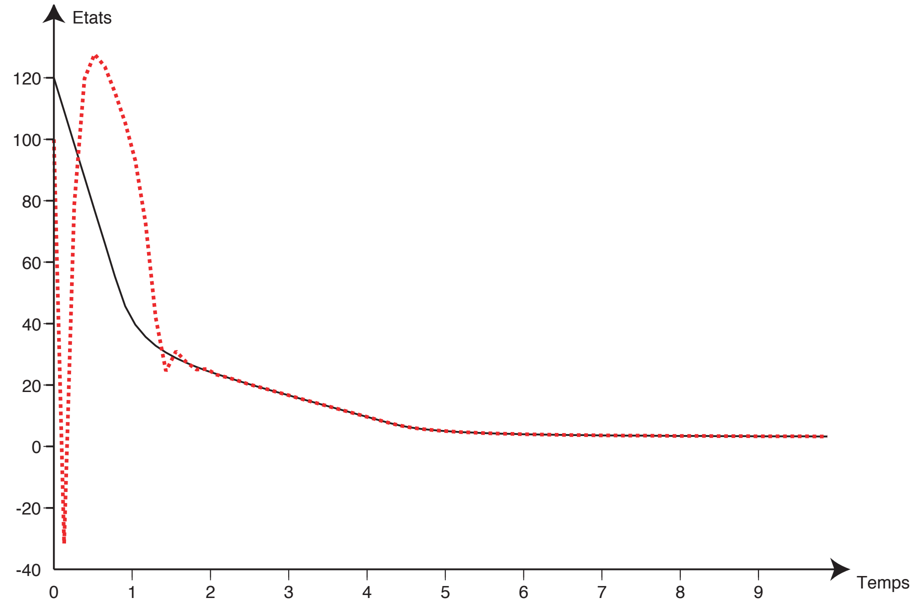

Figure 10: Simulation of system (17) with its observer (18): x_2 (solid line) and its estimate $\hat{x}_2$ (dashed line) when f is extended

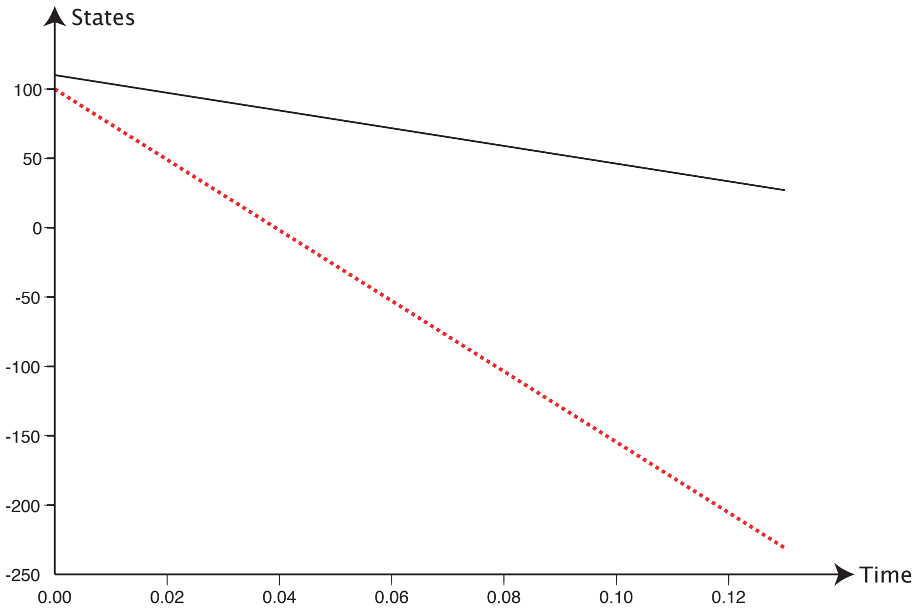

Figure 11: Simulation of system (17) with its observer (18): x_2 (solid line) and its estimate $\hat{x}_2$ (dashed line) when f is not extended

4 Conclusion

We have tried to combine modern Control Theory, Computer Science and Mathematics to address the state estimation problem for systems that model the dynamics of fish populations submitted to a fishing action. Indeed one of the important problems in fishery sciences is to estimate the state of the resource using the available data, in order to produce scientific opinions that can be helpful for developing management policies that need to have a good estimate of the available resource.
In this work, we have constructed High Gain observers for some fishery models. With the use of judicious value of the gain parameter θ we obtain satisfactory estimation of the real state. The observer's convergence is quite fast and does not depend on the initial conditions choice. Therefore one can get a "good" estimate of the unmeasurable real state very quickly. It is interesting to notice that the state estimator built in this paper for the stage-structured model use only the total catch to give not only an estimate of the total stock but also an estimate of the number of individuals in each stage class. The classical techniques like the Cohort Analysis (CA) or the Virtual Population Analysis (VPA) use the total catch for each stage-class in order to give estimates of the number of individuals in each stage class. In practice it is easier to measure the total catch 9without doing any distinction between individuals) then to measure the catch for each stage class. However the observers given in this paper assume that the model is good enough and that the parameters values are available.
Nonlinear control techniques are useful for studying and controlling complex systems. Although they have been initially developed for mechanical and electrical systems their applications to biological and environmental problems are growing. Tools of optimal control theory have been extensively used in renewable resource management ([8], [3] [23] [18] [25],[31],[45],[12],[32]). The present paper shows that the estimation problem in fisheries management can also be investigated from the point of view of control engineering.

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Appendix A. Positive invariance of D_w

Let $N = x_1 + x_2$.

$\dot{N} = -qEx_1 + r_1\left(1-\dfrac{x_1}{K_1}\right)x_1 + (N-x_1)\left(1-\dfrac{N-x_1}{K_2}\right)r_2$

Let w be a positive real number, for $N = w$, we have

$\dot{N} = -qEx_1 + r_1\left(1-\dfrac{x_1}{K_1}\right)x_1 + (w-x_1)\left(1-\dfrac{w-x_1}{K_2}\right)r_2 = g(x_1)$

The function g is defined for $0 \le x_1 \le w$.

$g(0) = w\left(1-\frac{w}{K_2}\right)r_2$

$g(w) = -quw + w\left(1-\frac{w}{K_1}\right)r_1$

$g'(x_1) = r_1 - r_2 - qu + \dfrac{2wr_2}{K_2} - 2\left(\dfrac{r_1}{K_1}+\dfrac{r_2}{K_2}\right)x_1$

$g'(x_1) = 0 \Leftrightarrow x_1 = \bar{x}_1 = \dfrac{K_1\left(K_2r_1 - K_2r_2 - quK_2 + 2wr_2\right)}{2\left(K_2r_1 + K_1r_2\right)}$

The maximum value of the function g is then given by the expression

$\dfrac{K_1K_2\left(qu - r_1 + r_2\right)^2 + \left(4K_2r_1r_2 + K_1\left(-4qur_2 + 4r_1r_2\right)\right)w - 4\left(r_1r_2\right)w^2}{4\left(K_2r_1 + K_1r_2\right)}$

It is therefore clear that this maximum is non positive if $w \ge w_0$ with

$$w_0 = \frac{r_1r_2(K_1+K_2) - quK_1r_2 + \sqrt{r_2\left(K_2r_1+K_1r_2\right)\left(K_1\left(-qu+r_1\right)^2 + K_2r_1r_2\right)}}{2r_1r_2} \tag{20}$$

This shows that for any real number $w \ge w_0$, the compact set

$$D_w = \{(x_1, x_2) \in \mathbb{R}_+^2 \ : \ x_1 + x_2 \le w\}$$

is positively invariant for system (17).

Appendix B. Construction of the Lipschitz extension of φ

The function φ is Lipschitz on the compact set D= $[a_0, b_0]\times[a_1, b_1]\times[a_2, b_2]$. Our aim is to extend it to a function $\tilde{\varphi}$ which is Lipschitz with the same Lipschitz coefficient in the whole $\mathbb{R}^3$.

Let $a(a_0, a_1, a_2)$, (respectively $b(b_0, b_1, b_2)$), the lower corner, (respectively the upper corner) of the domain D and $x(x_0, x_1, x_2)$ an unspecified point of $\mathbb{R}^3$.

The problem of the extension is set for point $x \notin$ D; in this situation we have 26 possibilities according to the situation of x. The different situations correspond to $x_i \le a_i$, $a_i \le x_i \le b_i$, or $x_i \ge b_i$.

The principle of this prolongation is to compose the function φ with the function π (the projection function of the point x on the domain D).

The extension of function φ is described by the following algorithm:

if $x_0 \le a_0$ then

  if $x_1 \le a_1$ then

if $x_2 \le a_2$ then
$\tilde{\varphi}(x_0, x_1, x_2) = \varphi(a_0, a_1, a_2)$
else
if $x_2 \le b_2$ then
$\tilde{\varphi}(x_0, x_1, x_2) = \varphi(a_0, a_1, x_2)$
else
$\tilde{\varphi}(x_0, x_1, x_2) = \varphi(a_0, a_1, b_2)$
end.
end.
else
if $x_1 \le b_1$ then
if $x_2 \le a_2$ then
$\tilde{\varphi}(x_0, x_1, x_2) = \varphi(a_0, x_1, a_2)$
else
if $x_2 \le b_2$ then
$\tilde{\varphi}(x_0, x_1, x_2) = \varphi(a_0, x_1, x_2)$
else
$\tilde{\varphi}(x_0, x_1, x_2) = \varphi(a_0, x_1, b_2)$
end.
end.
else
if $x_2 \le a_2$ then
$\tilde{\varphi}(x_0, x_1, x_2) = \varphi(a_0, b_1, a_2)$
else
if $x_2 \le b_2$ then
$\tilde{\varphi}(x_0, x_1, x_2) = \varphi(a_0, b_1, x_2)$
else
$\tilde{\varphi}(x_0, x_1, x_2) = \varphi(a_0, b_1, b_2)$
end.
end.
end.
end.
else
if $x_0 \le b_0$ then
if $x_1 \le a_1$ then
if $x_2 \le a_2$ then
$\tilde{\varphi}(x_0, x_1, x_2) = \varphi(x_0, a_1, a_2)$
else
if $x_2 \le b_2$ then
$\tilde{\varphi}(x_0, x_1, x_2) = \varphi(x_0, a_1, x_2)$
else
$\tilde{\varphi}(x_0, x_1, x_2) = \varphi(x_0, a_1, b_2)$
end.
end.
else
if $x_1 \le b_1$ then

            if $x_2 \leq a_2$ then
                $\tilde{\varphi}(x_0, x_1, x_2) = \varphi(x_0, x_1, a_2)$
            else
                if $x_2 \leq b_2$ then
                    $\tilde{\varphi}(x_0, x_1, x_2) = \varphi(x_0, x_1, x_2)$
                else
                    $\tilde{\varphi}(x_0, x_1, x_2) = \varphi(x_0, x_1, b_2)$
                end.
            end.
        else
            si $x_2 \leq b_2$ then
                $\tilde{\varphi}(x_0, x_1, x_2) = \varphi(x_0, b_1, x_2)$
            else
                $\tilde{\varphi}(x_0, x_1, x_2) = \varphi(x_0, b_1, b_2)$
            end.
        end.
    end.
else
    if $x_1 \leq a_1$ then
        if $x_2 \leq a_2$ then
            $\tilde{\varphi}(x_0, x_1, x_2) = \varphi(b_0, a_1, a_2)$
        else
            if $x_2 \leq b_2$ then
                $\tilde{\varphi}(x_0, x_1, x_2) = \varphi(b_0, a_1, x_2)$
            else
                $\tilde{\varphi}(x_0, x_1, x_2) = \varphi(b_0, a_1, b_2)$
            end.
        end.
    else
        if $x_1 \leq b_1$ then
            if $x_2 \leq a_2$ then
                $\tilde{\varphi}(x_0, x_1, x_2) = \varphi(b_0, x_1, a_2)$
            else
                if $x_2 \leq b_2$ then
                    $\tilde{\varphi}(x_0, x_1, x_2) = \varphi(b_0, x_1, x_2)$
                else
                    $\tilde{\varphi}(x_0, x_1, x_2) = \varphi(b_0, x_1, b_2)$
                end.
            end.
        else
            if $x_2 \leq a_2$ then
                $\tilde{\varphi}(x_0, x_1, x_2) = \varphi(b_0, b_1, a_2)$
            else
                if $x_2 \leq b_2$ then
                    $\tilde{\varphi}(x_0, x_1, x_2) = \varphi(b_0, b_1, x_2)$
                else

$\tilde{\varphi}(x_0, x_1, x_2) = \varphi(b_0, b_1, b_2)$
end.
end.
end.
end.
end.
end.